Phys 410
Spring 2013
Lecture #13 Summary
20 February, 2013

We considered the *driven* damped harmonic oscillator and resonance in detail. We take the driving function to be harmonic in time at a new frequency called simply ω, which is an independent quantity from the natural frequency of the un-damped oscillator, called ω_0. Employing the trick discussed in the last lecture, we want to solve this equation: $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. Note that the solution to the original problem can be found from $x(t) = Re[z(t)]$. We tried a solution of the form $z(t) = Ce^{i\omega t}$ and found this expression for the complex pre-factor: $C = \frac{f_0}{\omega_0^2 - \omega^2 + i2\beta\omega}$. We can write this complex quantity as a magnitude and phase as $C = Ae^{-i\delta}$, where A is the amplitude and δ is the phase, both real numbers. Solving for A and δ in terms of the oscillator parameters gives $A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}$, and $\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$. Finally, we can write the solution to the "z equation" as $z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}$.

The answer to the original problem is just the real part of this expression: $x(t) = Re[z(t)] = A\cos(\omega t - \delta)$, where ω is the frequency of the driving force. This represents the long-time persistent solution of the motion. It shows that the oscillator eventually adopts the same frequency as the driving force. In addition there is a solution to the homogeneous problem $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, which we solved before: $x_h(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2 - \omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}\ t}\right]$. The full solution is the sum of the particular solution and the homogeneous solution. In the case of small loss ($\beta < \omega_0$) the full solution can be written as $x(t) = A\cos(\omega t - \delta) + A_{tr}\, e^{-\beta t}\cos(\omega_1 t - \delta_{tr})$, where the first part is the particular solution and the second part is the transient (homogeneous) solution. We call it transient because of the $e^{-\beta t}$ factor, which shows that the initial motion and initial conditions (specified by A_{tr} and δ_{tr}) will eventually die off and the persistent motion will dominate.

The amplitude function $A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$ shows a resonant response. As a function of frequency ω at fixed natural frequency ω_0, there is a maximum amplitude of the persistent motion response when the driving frequency is equal to $\omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$. The quality factor of the resonance is a measure of how large and sharply peaked the amplitude response looks. It is defined as the ratio of the frequency at which there is peak energy (or power) amplitude over the frequency bandwidth known as the full-width at half maximum (FWHM). The FWHM is defined as the frequency width at the half-power height. The quality factor, or Q, is given by $Q = \omega_0/2\beta$.

The phase evolution through resonance goes from 0 well below resonance to π well above resonance, with $\delta = \pi/2$ exactly at resonance. The slope of $\delta(\omega)$ at resonance is $\frac{1}{\beta} = 2Q/\omega_0$.